Eur. Phys. J. B **61**, 331–334 (2008)
DOI: 10.1140/epjb/e2008-00069-1

THE EUROPEAN
PHYSICAL JOURNAL B

Upper and lower bounds for the speed of pulled fronts with a cut-off

R.D. Benguria[1], M.C. Depassier[1,a], and M. Loss[2]

[1] Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
[2] School of Mathematics, Georgia Tech, Atlanta, GA 30332, USA

Received 2 November 2007
Published online 20 February 2008 –

Abstract. We establish rigorous upper and lower bounds for the speed of pulled fronts with a cut-off. For all reaction terms of KPP type a simple analytic upper bound is given. The lower bounds however depend on details of the reaction term. For a small cut-off parameter the two leading order terms in the asymptotic expansion of the upper and lower bounds coincide and correspond to the Brunet-Derrida formula. For large cut-off parameters the bounds do not coincide and permit a simple estimation of the speed of the front.

PACS. 82.40.Ck Pattern formation in reactions with diffusion, flow and heat transfer – 52.35.Mw Nonlinear phenomena: waves, wave propagation, and other interactions – 02.30.Xx Calculus of variations

1 Introduction

The reaction diffusion equation

$$u_t = u_{xx} + f(u) \tag{1}$$

provides a simple description of phenomena in fields such as population dynamics, chemical reactions, flame propagation, fluids, QCD, among others [1–5]. It is one of the simplest models which shows how a small perturbation to an unstable state develops into a moving front joining a stable to an unstable state. The reaction term $f(u)$ satisfies different conditions depending on the physical problem of interest. One of the first, and most studied cases, is the Fisher reaction term $f(u) = u(1 - u)$ for which the asymptotic speed of the propagating front is $c = 2$, a value determined from linear considerations. A more general case was studied by Kolmogorov, Petrovskii and Piscounov (KPP)[6] who showed that for all reaction terms which satisfy the KPP condition

$$f(u) > 0, \qquad f(0) = f(1) = 0, \qquad f(u) < f'(0)u \tag{2}$$

the asymptotic speed of the front joining the stable $u = 1$ point to the unstable $u = 0$ point is given by

$$c_{KPP} = 2\sqrt{f'(0)}.$$

These fronts are called pulled since it is the leading edge of the front which determines the velocity of propagation. In the rest of this work we assume that $f'(0) = 1$. The evolution of localized initial conditions for general reaction terms, and rigorous properties of the fronts were studied by Aronson and Weinberger [7]. The asymptotic speed of the front for all reaction terms can be found from the integral variational principle [8]

$$c^2 = \sup_{g(u)} 2\,\frac{\int_0^1 f(u)g(u)d\,u}{\int_0^1 g^2(u)/h(u)d\,u} \tag{3}$$

where the supremum is taken over all positive monotonic decreasing functions $g(u)$ for which the integrals exist and where $h(u) = -g'(u)$. The supremum is always attained for reaction terms which are not pulled.

An effect not included in the classical reaction diffusion equation (1), is the effect of fluctuations. These arise when taking into account a finite number N of diffusive particles. It was shown by Brunet and Derrida that this effect can be simulated by the classical reaction equation by introducing a cut-off ϵ in the reaction term. The cut-off parameter ϵ is related to the number of diffusing particles by $\epsilon = 1/N$. It is also possible to model fluctuations by the explicit introduction of noise in the partial differential equation (1) in which case the cut-off parameter is proportional to the amplitude of noise.

By means of an asymptotic matching Brunet and Derrida showed that for a reaction term $f(u) = u(1 - u^2)$ a small cut-off changes the speed of the front to

$$c \approx 2 - \frac{\pi^2}{(\ln\epsilon)^2}. \tag{4}$$

In recent work it has been show that the Brunet-Derrida formula for the speed is correct to $\mathcal{O}((\ln\epsilon)^{-3})$ for a wider class of reaction terms [10].

[a] e-mail: mcdepass@gmail.com

The effect of a cut-off on the speed of pulled fronts has been studied extensively [9,11,12]. Of particular relevance to the present work are the results obtained in [13], where it is shown that the speed of the front is very sensitive to changes in the reaction term close to the cut-off point. The speed can be significantly greater or smaller than the KPP value $c = 2$. The results we present allow to assess the range of validity of the Brunet Derrida formula. We give rigorous bounds for the speed of the fronts which allow to estimate the speed for arbitrary reaction terms of KPP type and for all values of the cut-off parameter.

We show that for reaction terms of the form $\tilde{f}(u)\Theta(u-\epsilon)$ where $\tilde{f}$ satisfies the KPP condition equation (2) and Θ is the step function, the speed c of the front with the cut-off satisfies

$$2\sin(\phi_*) - \Delta(\phi_*) < c \le 2\sin(\phi_*), \tag{5}$$

with

$$\phi_* \tan(\phi_*) = \frac{1}{2}|\ln(\epsilon)|. \tag{6}$$

The upper bound $c = 2\sin\phi_*$ is the exact value of the speed for the reaction term shown with a solid line in Figure 1. The speed of fronts for all reaction terms of KPP type with a cut-off, depicted with the dotted line in Figure 1, is lower than this value.

We see that for $0 < \epsilon < 1$, $0 < \phi_* < \pi/2$. The function $\Delta(\phi_*)$ depends on the specific nonlinear terms of the reaction function. For small ϵ the series expansion of the upper bound c_{UP} is

$$c_{UP} = 2\sin(\phi_*) = 2 - \frac{\pi^2}{(\ln\epsilon)^2} + \mathcal{O}((\ln\epsilon)^{-3}). \tag{7}$$

The contribution of the nonlinearities, contained in the term $\Delta(\phi_*)$, appears at $\mathcal{O}((\ln\epsilon)^{-3})$, so that the leading order terms in the expansion of the upper and lower bounds give the Brunet-Derrida formula. In what follows we derive the bounds and apply them to the Fisher reaction term [14] $f(u) = u - u^2$ and to the reaction term studied by Brunet and Derrida $f(u) = u - u^3$. The main tool to obtain the bounds is the variational principle for the speed.

2 Upper and lower bounds

As shown in previous work [15], we may perform the change variables $u = u(s)$ where $s = 1/g$ in equation (3) and write the variational expression for the speed as

$$c^2 = \sup_{u(s)} 2\,\frac{F(1)/s_0 + \int_0^{s_0} F(u(s))/s^2 d\,s}{\int_0^{s_0} (du/ds)^2\, d\,s}, \tag{8}$$

where $s_0 = 1/g(u=1)$ is an arbitrary parameter,

$$F(u) = \int_0^u f(q)dq$$

and the supremum is taken over positive increasing functions $u(s)$ such that $u(0) = 0$, $u(s_0) = 1$ and for which all

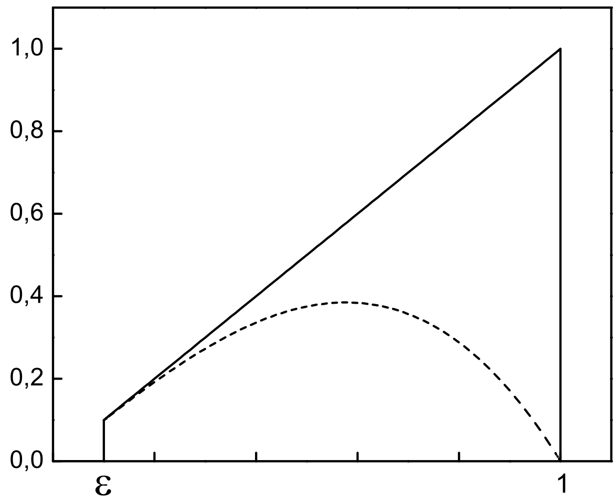

Fig. 1. The solid line shows the reaction term for which the exact speed can be determined. The dashed line is a typical reaction term of pulled type but with a cut-off.

the integrals in (8) are finite. Therefore, for any suitable trial function $u(s)$ we know that

$$c^2 \ge 2\,\frac{F(1)/s_0 + \int_0^{s_0} F(u(s))/s^2 d\,s}{\int_0^{s_0} (du/ds)^2\, d\,s}. \tag{9}$$

In the variational principle above if s_0 is finite then it can be scaled out of the problem. We have chosen to leave it explicit in order to allow for the possibility $s_0 = \infty$.

Consider now reaction terms $f(u)$ with a cut-off ϵ of the form

$$\tilde{f}(u)\Theta(u-\epsilon) = \begin{cases} 0 & \text{if } 0 \le u \le \epsilon \\ u - N(u) & \text{if } \epsilon < u < 1, \end{cases}$$

where $N(u)$, the nonlinearity, is such that $N(0) = N'(0) = 0$. We find

$$F(u) = \begin{cases} 0 & \text{if } 0 \le u \le \epsilon \\ u^2/2 - \epsilon^2/2 + F_n(u) & \text{if } \epsilon < u < 1, \end{cases}$$

where $F_n(u) = -\int_\epsilon^u N(u)du$.

Assume now that $\tilde{f}(u)$ satisfies the KPP criterion equation (2).

Since $\tilde{f}(u) < u$, it follows that $F(u) \le G(u)$ where

$$G(u) = \begin{cases} 0 & \text{if } 0 \le u \le \epsilon \\ u^2/2 - \epsilon^2/2 & \text{if } \epsilon < u < 1, \end{cases}$$

and therefore

$$c^2 < \mathcal{G}[u] \equiv \sup_{u(s)} 2\,\frac{G(1)/s_0 + \int_0^{s_0} G(u(s))/s^2 d\,s}{\int_0^{s_0} (du/ds)^2\, d\,s}. \tag{10}$$

Notice that $\mathcal{G}[u]$ is the speed of the reaction term depicted with a solid line in Figure 1. One can prove (rigorous details will be given elsewhere) that $\mathcal{G}$ is bounded above and that there exists a function $\hat{u}(s)$ for which the supremum is attained. This function is the monotonic increasing solution to the Euler-Lagrange equation for $\mathcal{G}$ satisfying the boundary conditions $\hat{u}(0) = 0, \hat{u}(s_0) = 1$. One can also prove that the variational parameter s_0 is finite and $\hat{u}'(s_0) = 0$. In summary, the maximizing function for $\mathcal{G}$ is the solution of

$$\frac{d^2\hat{u}}{ds^2} = 0 \qquad \text{for} \qquad 0 < \hat{u} < \epsilon$$

$$\frac{d^2\hat{u}}{ds^2} + \lambda \frac{\hat{u}}{s^2} = 0, \qquad \text{for} \qquad \epsilon < \hat{u} < 1$$

subject to the boundary conditions

$$\hat{u}(0) = 0, \qquad \hat{u}(s_0) = 1 \qquad \hat{u}'(s_0) = 0 \qquad \hat{u}'(s) > 0,$$

with the function and its derivative continuous at $\hat{u} = \epsilon$.

The solution to this problem is given by

$$\hat{u}(s) = \begin{cases} s & \text{if } 0 \le s \le \epsilon \\ A\sqrt{s}\cos(\phi(s)) & \text{if } \epsilon < s < s_0, \end{cases}$$

with

$$A = \sqrt{\epsilon}\sec(\phi_*), \qquad s_0 = 1/\epsilon, \tag{11}$$

and

$$\phi(s) = \frac{1}{2}\cot(\phi_*)\ln(s/\epsilon) - \phi_*, \tag{12}$$

where ϕ_* is the first positive solution of

$$\phi_* \tan\phi_* = \frac{1}{2}|\ln\epsilon|. \tag{13}$$

The maximum of $\mathcal{G} = \mathcal{G}[\hat{u}]$ can be calculated easily. We obtain after performing the integrals,

$$c^2 < \mathcal{G}[\hat{u}] = 4\sin^2(\phi_*) \equiv c^2_{UP}. \tag{14}$$

To obtain the lower bound we shall use the optimizing function $\hat{u}(s)$ as a a suitable trial function in equation (9). We obtain

$$c^2 \ge 4\sin^2(\phi_*) + \frac{4\sin(\phi_*)\cos^3(\phi_*)}{\epsilon(2\phi_* + \sin(2\phi_*))}\left[\epsilon F_n(1) + \int_\epsilon^{1/\epsilon} F_n(\hat{u}(s))/s^2 d\,s\right].$$

Since F_n is negative, we may combine equation (14) with the expression above and write our main result as given in equation (5).

3 Examples

As an example consider the reaction term studied by Brunet and Derrida, $\tilde{f}(u) = u - u^3$. The lower bound can be written explicitly as

$$c^2 > 4\sin^2(\phi_*) - \frac{2(1-\epsilon^2)\cos^3(\phi_*)\sin(\phi_*)}{2\phi_* + \sin(2\phi_*)} - \frac{2\epsilon\sin(\phi_*)}{\cos(\phi_*)(2\phi_* + \sin(2\phi_*))}\int_\epsilon^{1/\epsilon} \cos^4\phi(s)d\,s.$$

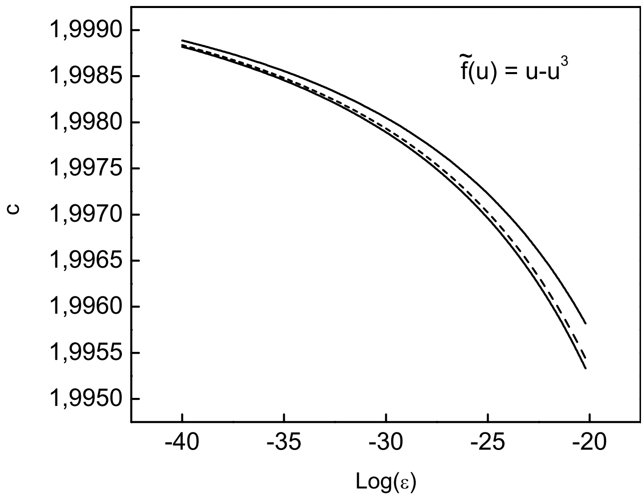

Fig. 2. Speed as a function of the cut-off parameter for the reaction term $\tilde{f}(u) = u - u^3$. The solid lines correspond to the bounds, the dashed line to the Brunet-Derrida formula.

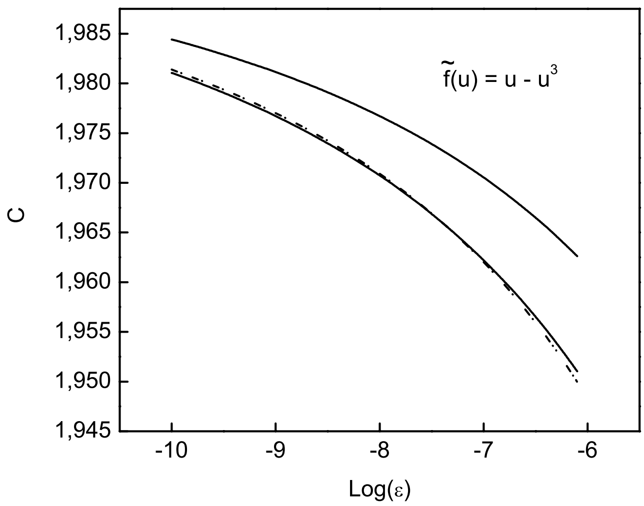

Fig. 3. As in Figure 2 for different values of the cut-off parameter.

The integral has a long analytic expression which we omit here. From the explicit expression above it is not difficult to show that the contribution of the two last terms, which arise from the nonlinear terms, are of $\mathcal{O}(|\ln\epsilon|^{-3}|$. In Figures 1 and 2 we show the bounds together with the Brunet-Derrida formula as a function of ϵ. The solid lines correspond to the upper and lower bounds. The dashed line is the Brunet-Derrida formula.

As a second example we consider the Fisher reaction term $\tilde{f}(u) = u(1-u)$ with a cut-off. The lower bound becomes

$$c^2 > 4\sin^2(\phi_*) - \frac{8}{3}\frac{(1-\epsilon)\sin(\phi_*)\cos^3(\phi_*)}{(2\phi_* + \sin(2\phi_*))} - \frac{8}{3}\frac{\sqrt{\epsilon}\sin(\phi_*)}{(2\phi_* + \sin(2\phi_*))}\int_\epsilon^{1/\epsilon}\frac{\cos^3\phi(s)}{\sqrt{s}}d\,s.$$

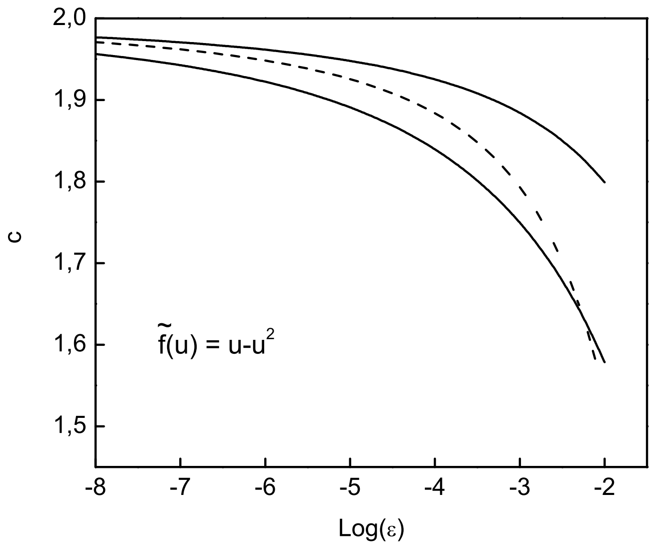

Fig. 4. Speed as a function of the cut-off parameter for the reaction term $\tilde{f}(u) = u - u^2$. Lines as in Figure 2.

Again, the integral can be done analytically and we do not show it here.

In Figure 4 we show the upper and lower bounds and the Brunet-Derrida formula. In this case the Brunet-Derrida formula leaves the allowed band at larger value of ϵ. In general for reaction terms $\tilde{f}(u) = u - u^n$, the gap between the upper and lower bounds becomes narrower and the Brunet-Derrida formula valid for a smaller range of ϵ.

4 Summary

In summary, we have studied the effect of a cut-off on reaction terms which satisfy the KPP condition equation (2). We have found upper and lower bounds valid for all values of the cut-off parameters, which allow to assess the accuracy of the Brunet Derrida formula, and to estimate the speed of the front. If we consider only the piecewise linear reaction term $\tilde{f}(u) = u$, $\tilde{f}(1) = 0$, the upper and lower bounds coincide and give the exact value for the speed, of which the two leading order terms are the Brunet-Derrida formula.

We acknowledge partial support of Fondecyt (CHILE) projects 106–0627 and 106–0651, CONICYT/PBCT Proyecto Anillo de Investigación en Ciencia y Tecnología ACT30/2006 and U.S. National Science Foundation grant DMS 06-00037.

References

1. J.D. Murray, *Mathematical Biology* (Springer Verlag, New York, 1993)
2. N.F. Britton, *Reaction-Diffusion Equations and Their Applications to Biology* (Academic Press, London, 1986)
3. K. Showalter, Nonlinear Science Today **4**, 1 (1995)
4. Ya. B. Zeldovich, S.B. Pikelner, Soviet Physics JETP **29**, 170 (1969)
5. S. Munier, R. Peschanski, Phys. Rev. Lett. **91**, 232001 (2003)
6. A.N. Kolmogorov, I.G. Petrovskii, N.S. Piskunov, *Selected Works of A.N. Kolmogorov*, edited by V.M. Tikhomirov (Kluwer Academic Publishers, 1991)
7. D.G. Aronson, H.F. Weinberger, Adv. Math. **30**, 33 (1978)
8. R.D. Benguria, M.C. Depassier, Phys. Rev. Lett. **77**, 1171 (1996)
9. E. Brunet, B. Derrida, Phys. Rev. E **56**, 2597 (1997)
10. F. Dumortier, N. Popovic, T.J. Kaper, Nonlinearity **20**, 855 (2007)
11. D. Panja, Phys. Rep. **393**, 87 (2004)
12. V. Mendez, D. Campos, E.P. Zemskov, Phys. Rev. E **72** 056113 (2005)
13. D. Panja, W. van Saarloos, Phys. Rev. E **66**, 015206 (2002)
14. R.A. Fisher, Ann. Eugenics **7**, 353 (1937)
15. R.D. Benguria, M.C. Depassier, Phys. Rev. E **75**, 051106 (2007)